SUMMER-2020

Q.1 (a) If $A=\{a,b\}$ and $B=\{c,d\}$ and $C=\{e,f\}$ then find (i) $(A\times B)\cup(B\times C)$ (ii) $A\times(B\cup C)$.

Solⁿ:-

(i) $A=\{a,b\}$, $B=\{c,d\}$, $C=\{e,f\}$

$A\times B=\{(a,c),(a,d),(b,c),(b,d)\}$

$B\times C=\{(c,e),(c,f),(d,e),(d,f)\}$

$(A\times B)\cup(B\times C)=\{(a,c),(a,d),(b,c),(b,d),(c,e),(c,f),(d,e),(d,f)\}$

(ii) $B\cup C=\{c,d,e,f\}$

$A\times(B\cup C)=\{(a,c),(a,d),(a,e),(a,f),(b,c),(b,d),(b,e),(b,f)\}$

(b) Define even and odd functions. Determine whether the function $f: I\to R^+$ defined by $f(x)=2x+7$ is one-to-one or bijective.

Solⁿ:-

Given that $f: I\to R^+$ defined by

$f(x)=2x+7$

let $f(x_1)=f(x_2)$

$2x_1+7=2x_2+7$

$2x_1=2x_2$

$x_1=x_2$

$\therefore f: I\to R^+$ is one one

Again $f(x)=2x+7$

$y=2x+7$

$x=\frac{y-7}{2}$

For $y=5$ we have $x=-1 \neq 1$

$\therefore f(x)=2x+7$ is not onto

$\therefore f(x)$ is one one only.

(c) (i) Show that the relation $x \equiv y \pmod m$ defined on the set of integers I is an equivalence relation.

(ii) Draw the Hasse diagram for the partial ordering $\{(A,B) \mid A \subseteq B\}$ on the power set $P(S)$, where $S=\{a,b,c\}$.

Solⁿ:- Given that the relation is $x \equiv y \pmod m$

$\therefore (x-y)$ is divisible by m

$(x-y)=mk, \; k \in I$

i.e. $xRy : (x-y)=mk, \; k \in I$

Reflexive : For $\forall x \in I$

$(x-x)=0$

$(x-x)=mk, \; k=0 \in I$

xRx

$\therefore R$ is Reflexive.

Symmetric : For $\forall x,y \in I, \; xRy$

$(x-y)=mk$

$y-x=-mk$

$(y-x)=m(-k) \quad -k \in I$

$\therefore yRx$

$\therefore R$ is symmetric

Transitive : Let xRy and yRz

$(x-y)=mk_1$ and $(y-z)=mk_2, \; k_1, k_2 \in I$

$(x-y)+(y-z)=m(k_1+k_2)$

$(x-z) = m(k_1 + k_2)$; $k_1 + k_2 \in I$

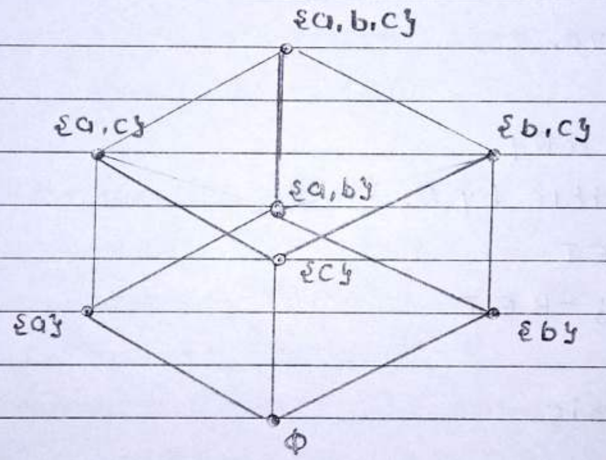
$\therefore xRz$

$\therefore$ R is Transitive.

$\therefore$ The relation R on I defined by $x \equiv y \pmod{m}$ is an equivalence relation.

(ii) $S = \{a, b, c\}$

$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$.



Here/Sr

Q.2(a) define equivalence class. Let R be the relation on the set of integers I defined by $(x-y)$ is an even integer, find the disjoint equivalence classes.

Sol^n:- Let R be the relation in the integers I defined by $(x-y)$ is an even integers i.e. $(x-y)$ is divisible by 2.

Reflexive: For all $x \in I$ we have $(x-x)=0$
i.e. $(x-x)=2K$, $K=0 \in I$
i.e. xRx $\forall x \in I$
$\therefore$ R is Reflexive.

Symmetric: Let xRy
i.e $x-y$ is divisible by 2.
$\therefore (x-y)=2K$, $K \in I$
$(y-x)=2(-K)$; $-K \in I$
i.e. yRx
$\therefore$ R is symmetric

Transitive: Let xRy and yRz
i.e $(x-y)$ and $(y-z)$ are even integers
$\therefore (x-y)=2K_1$ and $(y-z)=2K_2$; $K_1, K_2 \in I$
$\therefore (x-y)+(y-z)=2(K_1+K_2)$; $K_1+K_2 \in I$
$\therefore (x-z)$ is an even integer
$\therefore xRz$
$\therefore$ R is transitive.
hence the relation R is equivalence relation.
The disjoint equivalence classes are
$[0]=\{\ldots -4,-2,0,2,4,6 \ldots\}$
$[1]=\{\ldots -3,-1,1,3,5,7 \ldots\}$

(b) A committee of 5 persons is ~~no~~ to be formed from 6 men and 4 women. In how many ways can this be done when (i) at least 2 women are included (ii) at most 2 women are included?

Soln:- i) at least 2 women are included.

The committee may consist of

3 women, 2 men : $^4C_3 \times {}^6C_2 = 4 \times 15 = 60$

4 women, 1 men : $^4C_4 \times {}^6C_1 = 1 \times 6 = 6$

2 women, 3 men : $^4C_2 \times {}^6C_3 = 6 \times 20 = 120$

$\therefore$ Total number of ways of forming the committee =

$60 + 6 + 120 = 186$

ii) when at most 2 women are included

2 women, 3 men : $^4C_2 \times {}^6C_3 = 6 \times 20 = 120$

1 women, 4 men : $^4C_1 \times {}^6C_4 = 4 \times 15 = 60$

5 men : $^6C_5 = 6 = 6$

$\therefore$ Total number of ways of forming the committee =

$120 + 60 + 6 = 186.$

Q.(c) Solve the recurrence relation $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2$ using the method of undetermined coefficients.

Sol^n:- (a) Homogeneous Solution:

The characteristic equation is

$\alpha^2 + 5\alpha + 6 = 0 \Rightarrow \alpha = -2, -3.$

$\therefore a_n^{(h)} = A_1(-2)^n + A_2(3)^n$

(b) Particular Solution:

We have $f(n) = 3n^2$

$\therefore$ Guess $a_n^{(p)} = A_0 + \cancel{A_n}\, A_1 n + A_2 n^2$

putting it in the given relation, we get

$A_0 + A_1 n + A_2 n^2 + 5(A_0 + A_1(n-1) + A_2(n-1)^2)$
$+ 6(A_0 + A_1(n-2) + A_2(n-2)^2) = 3n^2$

$\Rightarrow A_0 + A_1 n + A_2 n^2 + 5A_0 + 5nA_1 - 5A_1 + 5n^2A_2 - 10nA_2 + 5A_2$
$+ 6A_0 + 6nA_1 - 12A_1 + 6n^2A_2 - 24nA_2 + 24A_2 = 3n^2.$

$\Rightarrow (12A_0 - 17A_1 + 29A_2) + (12A_1 - 34A_2)n + 12n^2A_2 = 3n^2$

Comparing on both sides, we get

$12A_2 = 3,\ 12A_1 - 34A_2 = 0,\ 12A_0 - 17A_1 + 29A_2 = 0$

$\Rightarrow A_2 = 1/4,\ A_1 = 17/24,\ A_0 = 115/288$

$$\therefore a_n^{(p)} = \frac{115}{288} + \frac{17}{24}n + \frac{1}{4}n^2$$

$\therefore$ The general solution is

$a_n = a_n^{(h)} + a_n^{(p)}$

$$= A_1(-2)^n + A_2(3)^n + \frac{115}{288} + \frac{17}{24}n + \frac{1}{4}n^2.$$

(c) Solve the recurrence relation using the method of generating function $a_n - 5a_{n-1} + 6a_{n-2} = 3^n, n \geq 2; a_0 = 0, a_1 = 2$.

Sol^n:- Multiplying on both sides of the given relation by x^n and summing from $n=2$ to ∞, we get

$$\sum_{n=2}^{\infty} a_n x^n - 5\sum_{n=2}^{\infty} a_{n-1} x^n + 6\sum_{n=2}^{\infty} a_{n-2} x^n = \sum_{n=2}^{\infty} 3^n x^n \quad \ldots (1)$$

Also, we know that $G(x) = \sum_{n=0}^{\infty} a_n x^n$. Thus (1) becomes

$$\{a_2 x^2 + a_3 x^3 + \ldots\} - 5\{a_1 x^2 + a_2 x^3 + \ldots\} + 6\{a_0 x^2 + a_1 x^3 + \ldots\} = \sum_{n=2}^{\infty} 3^n x^n$$

$$\therefore \{G(x) - a_0 - a_1 x\} - 5x\{G(x) - a_0\} + 6x^2 G(x) = 3^2 x^2 \sum_{n=2}^{\infty} 3^{n-2} x^{n-2}$$

$$\therefore G(x)\{1 - 5x + 6x^2\} + a_0(-1 + 5x) + a_1\{-x\} = 3^2 x^2 \sum_{k=0}^{\infty} 3^k x^k$$

(put $k = n-2$)

$$\therefore G(x)(1-2x)(1-3x) - 2x = \frac{9x^2}{1-3x}.$$

$$\therefore G(x)(1-2x)(1-3x) = \frac{9x^2}{1-3x} + 2x = \frac{3x^2 + 2x}{1-3x}$$

$$\therefore G(x) = \frac{3x^2 + 2x}{(1-2x)(1-3x)^2}$$

$$= \frac{A}{1-2x} + \frac{B}{1-3x} + \frac{C}{(1-3x)^2}$$

$\therefore 3x^2 + 2x = A(1-3x)^2 + B(1-2x)(1-3x) + C(1-2x)$

equating coefficients of x^2, x and 1, we get

$3 = 9A + 6B$, $2 = -6A - 5B - 2C$, $0 = A + B + C$

Solving, we get $A = 7$, $B = -10$, $C = 3$

$$\sum_{n=0}^{\infty} a_n x^n = \frac{7}{1-2x} - \frac{10}{1-3x} + \frac{3}{(1-3x)^2}$$

$$= 7\sum_{n=0}^{\infty} 2^n x^n - 10\sum_{n=0}^{\infty} 3^n x^n + 3\sum_{n=0}^{\infty}(n+1)3^n x^n$$

$\therefore a_n = 7\cdot(2)^n - 10(3)^n + 3(n+1)(3)^n$

$\therefore a_n = 7(2)^n + (3n-7)(3)^n$ is the required solution.

Q.3 (a) define simple graph, degree of a vertex and complete graph.

Soln:- Simple graph:- The graph free from self loops and parallel edges is called a simple graph.

Degree of a vertex: The number of edges incident with a vertex is called the degree of a vertex. Each self loop is counted twice. It is denoted by $deg(v)$.

Complete graph: The simple graph is said to be a complete or full graph if there exist an edge between each and every pair of vertices. A complete graph of n vertices is denoted by K_n.

(b) Define tree. Prove that there is one and only one path between every pair of vertices in a tree T.

Soln:- A connected graph without any circuit is called a tree. It is denoted by T.

Since T is connected there always exist at least one path between every pair (v_i, v_j) of vertices. If there is another path between (v_i, v_j) then the union of such two paths forms a circuit, which is not true as T is free from circuit. !

(c) (i) A graph G has 15 edges, 3 vertices of degree 4 and other vertices of degree 3. Find the number of vertices in G.

(ii) Define vertex disjoint and edge disjoint subgraphs by drawing the relevant graphs.

Solⁿ:-

i) Let n be the number of vertices in G

$\therefore$ The degree of $G = 2(15) = 30$

Also, the degree of $G = 3(4) + (n-3)3$

$$\therefore 12 + 3n - 9 = 30$$

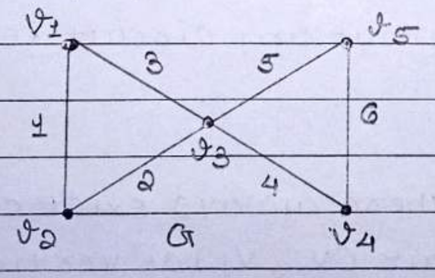
$$3n = 27$$

$$n = 9$$

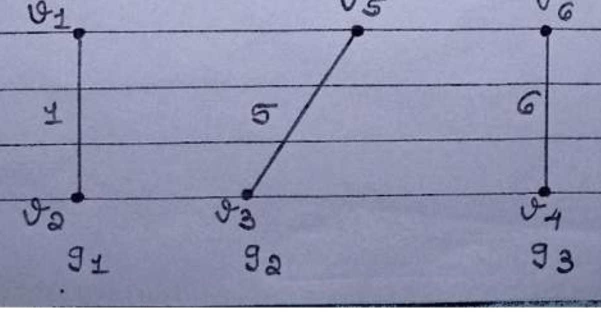
$\therefore$ Total number of vertices in G is 9.

ii) Vertex disjoint subgraphs: Two or more subgraphs of a graph G are said to be vertex disjoint if they do not have vertices in common.

For example, consider.



Edge disjoint subgraphs: Two or more subgraphs of a graph G are said to be edge disjoint if they do not have edge in common.



Q.4 (a) define algebric structure, semi group and monoid. Also give related examples.

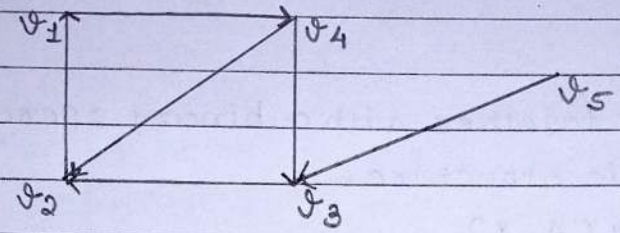
Sol^n:-

A non empty set A together with a binary operation * is called an algebraic structure.
It is denoted by (A, *).

An algebric system (A, *) is called a semigroup if it satisfies the following properties.
1. A is closed with respect to *
2. * is associative
e.g. Set A of all positive even integers and * is a binary operation of multiplication then
1. $\forall a, b \in A$ we have $a * b = ab$ = even number
$\therefore a * b \in A$
Thus A is closed under *
2. $\forall a, b, c \in A$ we have
$(a * b) * c = (ab) * c = abc$
and $a * (b * c) = a * (bc) = abc$
$\therefore (a * b) * c = a * (b * c)$
$\therefore$ * is associative
$\therefore (A, *)$ is a semi group.

Monoid: An algebric structure (A, *) is called a monoid if it satisfies the following properties.
1. * is closed on A
2. * is associative of A
3. There exists an identity element with respect to *.

(b) Use Warshall's algorithm to obtain path matrix from the adjacency matrix of



Sol^n:- The adjacency matrix is

$$M = \begin{array}{c|ccccc|} & v_1 & v_2 & v_3 & v_4 & v_5 \\ v_1 & 0 & 0 & 0 & 1 & 0 \\ v_2 & 1 & 0 & 0 & 0 & 0 \\ v_3 & 0 & 1 & 0 & 0 & 0 \\ v_4 & 0 & 1 & 1 & 0 & 0 \\ v_5 & 0 & 0 & 1 & 0 & 0 \end{array}$$

$W_0 = M$

$$W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

For W_2 ($K=2$) in second column R_3 and R_4 have 1. In second row C_1 and C_4 have 1. Thus put 1 at (R_3, C_1), (R_3, C_4), (R_4, C_1) and (R_4, C_4)

$$W_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

For W_3 (K=3), In third column, R_4 and R_5 have 1. In third row C_1, C_2, C_4 have 1. Thus put 1 at (R_4, C_1), (R_4, C_2), (R_4, C_4), (R_5, C_1), (R_5, C_2), (R_5, C_4).

$$W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

For W_4 (K=4) In fourth column R_1, R_2, R_3, R_4, R_5 have 1. In fourth row C_1, C_2, C_3, C_4 have 1. Thus above put 1 at the new pairs.

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

For W_5 (K=5) In fifth column there is no row containing 1. So W_5 is same as W_4.

$W_5 = W_4 \doteq$ Path matrix.

(c) (i) Is the algebraic system $(Q, *)$ a group? where Q is the set of rational numbers and $*$ is a binary operation defined by $a*b = a+b-ab$, $\forall a,b \in Q$.

(ii) Let $(Z, +)$ be a group, where Z is the set of integers and $+$ is an operation of addition. Let H be a subgroup of Z consisting of elements multiple of 5. Find the left cosets of H in Z.

Soln:- Let $a,b,c \in Q$.

Since, $a,b \in Q$ $a+b-ab \in Q$

Thus $a*b \in Q$

$\therefore Q$ is closed under $*$.

$(a*b)*c = (a+b-ab)*c$

$= (a+b-ab)+c-(a+b-ab)c$

$= a+b-ab+c-ac-bc+abc$

and $a*(b*c) = a*(b+c-bc)$

$= a+(b+c-bc)-a(b+c-bc)$

$= a+b+c-bc-ab-ac+abc$

$\therefore (a*b)*c = a*(b*c)$

$\therefore Q$ is associative under $*$.

For $a*e = a \Rightarrow a+e-ae = a$

$\Rightarrow e(1-a) = 0 \Rightarrow e = 0$ when $a \neq 1$

$\therefore e = 0 \in Q$ is an identity element.

For $a*b = e \Rightarrow a+b-ab = 0$

$\Rightarrow b(1-a) = -a \Rightarrow b = \frac{a}{a-1}$, $a \neq 1$.

$\therefore$ Each element has an inverse such that $a \neq 1$.

$\therefore (Q, *)$ is not a group as 1 has no inverse.

ii) we have $H = \{\ldots -10, -5, 0, 5, 10 \ldots\}$

Left coset are:

$0+H = \{\ldots -10, -5, 0, 5, 10 \ldots\}$

$1+H = \{\ldots -9, -4, 1, 6, 11 \ldots\}$

$2+H = \{\ldots -8, -3, 2, 7, 12 \ldots\}$

$3+H = \{\ldots -7, -2, 3, 8, 13 \ldots\}$

$4+H = \{\ldots -6, -1, 4, 9, 14 \ldots\}$

$5+H = \{\ldots -5, 0, 5, 10, \ldots\} = 0+H$

$6+H = \{\ldots -4, 1, 6, 11 \ldots\} = 1+H$ etc.

Q.5 (a) Show that the operation $*$ defined by $x * y = x^y$ on the set N of natural numbers is neither commutative nor associative.

Solⁿ:- Since $3 * 4 = 3^4$ and $4 * 3 = 4^3$ are different, $*$ is not commutative.

Now $(2 * 3) * 4$ and $2 * (3 * 4) = 2 * 3^4$

$= 2^3 * 4$ $\qquad = 2^{3^4}$

$= (2^3)^4$ $\qquad = 2^{81}$

$= 2^{12}$

$\therefore$ $*$ is not associative.

(b) Prove that an algebraic structure $(G, *)$ is an abelian group, where G is the set of non zero real numbers and $*$ is a binary operation defined by $a * b = \frac{ab}{2}$.

Solⁿ:- Let $a, b, c \in G$

$a * b = \frac{ab}{2}$ is a non-zero real number.

$\therefore a * b \in G$. Thus G is closed under $*$.

$(a * b) * c = (ab/2) * c$ $\qquad a * (b * c) = a * (bc/2)$

$= \frac{(ab/2)c}{2}$ $\qquad = \frac{a(bc/2)}{2}$

$= abc/4$ $\qquad = abc/4$

$\therefore (a * b) * c = a * (b * c)$

$\Rightarrow$ G is associative under $*$.

For $a * e \Rightarrow a \Rightarrow \frac{ae}{2} = a \Rightarrow e = 2 \in G$ is an identity element

For $a * b = e \Rightarrow \frac{ab}{2} = 2 \Rightarrow b = \frac{4}{a} \in G$ is an inverse of a.

$a * b = \frac{ab}{2} = \frac{ba}{2} = b * a$

$\therefore G$ is commutative under $*$.

$\therefore (G, *)$ is an abelian group.

(c) (i) Find out using truth table, whether $(p \wedge r) \rightarrow p$ is a tautology.

(ii) Obtain the dnf of the form $\sim(p \rightarrow (q \wedge r))$.

Soln:- (i)

p	r	$p \wedge r$	$(p \wedge r) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Since the truth value of $(p \wedge r) \rightarrow p$ is T for all value of p and r, the proposition is a tautology.

(ii) $\sim(p \rightarrow (q \wedge r))$

$\equiv \sim(\sim p \vee (q \wedge r))$

$\equiv \sim(\sim p \vee (q \wedge r))$

$\equiv \sim(\sim p) \wedge \sim(q \wedge r)$

$\equiv p \wedge (\sim p \vee \sim r)$

$\equiv (p \wedge \sim q) \vee (p \wedge \sim r)$.