WINTER-2020

Q.1 (a) Find the power sets of i) {a} ii) {a,b,c}.

Ch:-1

Soln:-

i) Let $A = \{a\}$

$P(A) = \{A, \phi\}$.

ii) Let $S = \{a,b,c\}$

$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$.

(b) If $f(x) = 2x$, $g(x) = x^2$, $h(x) = x+1$ then find $(fog)oh$ and $fo(goh)$

Soln:-

$(fog)oh$

$= fog(h(x))$

$= fog(x+1)$

$= f[g(x+1)]$

$= f[x^2(x+1)]$

$= f(x^3+x^2)$

$= 2(x^3+x^2)$

$= 2x^3+2x^2$

$fo(goh) = f(goh)(x) = f(g(h(x)))$

$= f(g(x+1))$

$= f((x+1)^2)$

$= 2(x+1)^2$.

(i)
(c) Let N be the set of natural numbers. Let R be a relation in N defined by $x R y$ if and if $x+3y=12$. Examine the relation for (i) reflexive (ii) symmetric (iii) transitive.

Solⁿ:- Given that R is relation in N defined by

$x R y : x+3y=12$

Reflexive : Let $x+3y=12$ for $y=x$

$\therefore 4x=12$

$x=3$

$\therefore x R x$ for $x=3$ only

But $x R x \ \forall x \in N$

hence R is not reflexive.

Symmetric : Let $x R y$

i.e $x+3y=12$

i.e $y+3y$ may or may not equal to 12

$\therefore y \not R x$

hence R is not symmetric.

Transitive : Let $x R y$ and $y R z$

$\therefore x+3y=12$ and $y+3z=12$

This holds $x=y=z=3 \in N$

i.e. $x+3z=12$

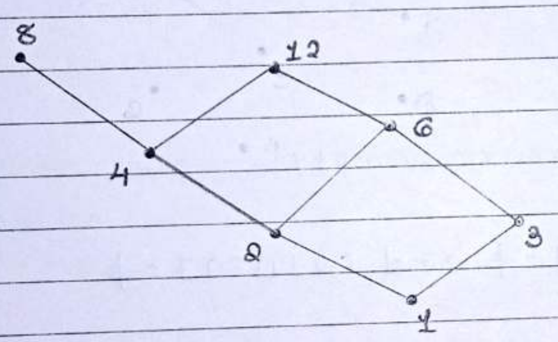
$x R z$

$\therefore$ R is transitive.

(ii) Draw the Hase diagram representing the partial ordering {(a,b) | a divides b} on {1,2,3,4,6,8,12}.

Sol^n:- R = {(1,1), (2,2), (3,3), (4,4), (6,6), (8,8), (12,12), (1,2), (1,3), (1,4), (1,6), (1,8), (1,12), (2,4), (2,6), (2,8), (2,12), (3,6), (3,12), (4,8), (4,12), (6,12)}

hase diagram.



Q.2 (a) Let R be a relation defined in $A = \{1, 2, 3, 5, 7, 9\}$ as $R = \{(1,1), (1,3), (1,5), (1,7), (2,2), (3,1), (3,3), (3,5), (3,7), (5,1), (5,3), (5,5), (7,1), (7,3), (7,5), (7,7), (9,9)\}$. Find the partitions of A based on the equivalance relation R.

Solⁿ:- Let $A = \{1, 2, 3, 5, 7, 9\}$

$R = \{(1,1), (1,3), (1,5), (1,7), (2,2), (3,1), (3,3), (3,5), (3,7), (5,1), (5,3), (5,5), (7,1), (7,3), (7,5), (7,7), (9,9)\}$

The disjoint equivalence classes are

$[1] = \{1, 3, 5, 7\}$

$[9] = \{9\}$

$[2] = \{2\}$

obviously

i) The set $[1]$, $[2]$ and $[9]$ are non empty

ii) $[1] \cap [2] = \phi$; $[1] \cap [9] = \phi$ and $[2] \cap [9] = \phi$

iii) $[1] \cup [2] \cup [9] = A$

hence $[1], [2], [9]$ is a partition of A.

(b) In a box there are 5 black pens, 3 white pens and 4 red pens. In how many ways can 2 black pens, 2 white pens and 2 red pens can be chosen?

Solⁿ:- Number of ways of choosing 2 black pens from 5 black pens in 5_{C_2} ways.

Number of ways of choosing 2 white pens from ~~pens~~ 3 white pens in 3_{C_2} ways.

Number of ways of choosing 2 red pens from 4 red pens in $4C_2$ ways.

By the counting priniciple, 2 black, 2 white and 2 red pens can be chosen in

$= 5_{C_2} \times 3C_2 \times 4C_2$

$= 10 \times 3 \times 6$

$= 180$ ways.

(c) Solve the recurrence relation $a_n - 4a_{n-1} + 4a_{n-2} = n + 3^n$ using undetermined coefficient method.

Sol^n:- (a) homogeneous solution:

$a_n - 4a_{n-1} + 4a_{n-2} = n + 3^n$

The characteristic equation is

$\alpha^2 - 4\alpha + 4 = 0$

$\Rightarrow \alpha = 2, 2$

$\therefore a_n^{(h)} = (A_0 + A_1 \cdot n)2^n$

(b) Particular solution:

we have $f(n) = n + 3^n$ (n is a polynomial of degree 1 and $\beta = 3$ is not a root of the characteristic equation)

$\therefore$ Guess

$$a_n^{(p)} = (A_0 + A_1 \cdot n) + A_2 \cdot 3^n$$

putting it in the given relation, we get

$$A_0 + A_1 \cdot n + A_2 \cdot 3^n - 4(A_0 + A_1 \cdot (n-1) + A_2 \cdot 3^{n-1}) + 4(A_0 + A_1 \cdot (n-2) + A_2 \cdot 3^{n-2}) = n + 3^n.$$

$$\therefore A_0 + nA_1 + 3^n A_2 - 4A_0 - 4nA_1 - 4A_1 - 4A_2 3^{n-1} + 4A_0 + 4nA_1 - 8A_1 + 4A_2 3^{n-2} = n + 3^n.$$

$$\therefore A_0 - 4A_1 + nA_1 + 3^n A_2 - 4A_2 3^{n-1} + 4A_2 3^{n-2} = n + 3^n$$

$$\therefore (A_0 - 4A_1) + nA_1 + 3^n\left(A_2 - \frac{4}{3}A_2 + \frac{4}{9}A_2\right) = n + 3^n$$

Comparing on both sides, we get

$A_0 - 4A_1 = 0, \ A_1 = 1, \ A_2 - \frac{4}{3}A_2 + \frac{4}{9}A_2 = 1$

$\therefore A_0 + 4, \ A_1 = 1, \ A_2 = 9$

$\therefore a_n^{(P)} = 4 + n + 9 \cdot 3^n$

$= 4 + n + 3^{n+2}$

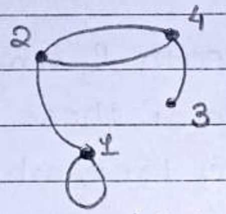
$\therefore$ The general solution is

$a_n = a_n^{(h)} + a_n^{(P)}$

$= (A_0 + nA_1)2^n + 4 + n + 3^{n+2}.$

Q.3 (a) Define self-loop, adjacent vertices and a pendant vertex.

Sol^n:- • Self-loop is an edge that connects a vertex to itself.



A group with a self-loop on vertex 1.

• If two vertices in a graph are connected by an edge, we say the vertices are adjacent.

• Let G be a graph, A vertex v of G is called a pendant vertex if and only if v has degree 1. In other words, pendant vertices are the vertices that have degree 1, also called pendant vertex.

(b) Define tree. Prove that if a graph G has one and only one path between every pair of vertices then G is a tree.

Sol^n:- A tree is a connected acyclic graph. In other words a simple connected graph without circuits is known as tree.

Since there exist a path between every two vertices of G this shows that G is a connected graph. Also G cannot contain a circuit as these paths are unique (one and only one) hence G is a connected graph without circuit.
Therefore G is a tree.

Conversely, if the graph G is a tree, it is connected and hence, there must be atleast one path between every pair of vertices in G.

Suppose there are two distinct path P_1 and P_2 between vertices a and b of G. The union of these paths will contain a circuit and then G will not be a tree, which is a contradiction.

Therefore, if G is a tree it has a one and only one path between every pair of vertices of G.

(c) (i) Find the number of edges in G if it has 5 verties each of degree 2.

(ii) Define complement of a subgroup by drawing the graphs.

Sol^n:- Let e be the number of edges in G.

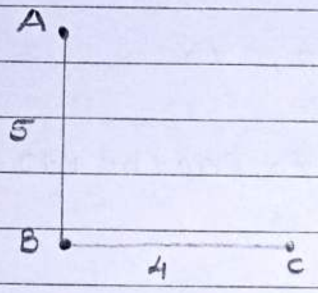
$\therefore$ The degree of $G = 2e$

$\therefore 2(5) = 2e$

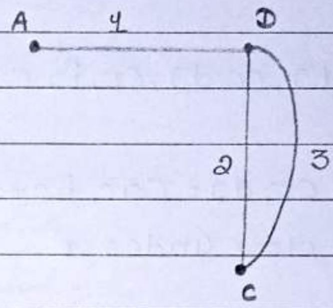
$\Rightarrow e = 5$

Let $G = (V, E)$ be a graph and $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two subgraphs of G. Then G_2 is called a complement of G_1 when $E_2 = E - E_1$ and V_2 contains only the verties of V which are incident with edges in E_2.

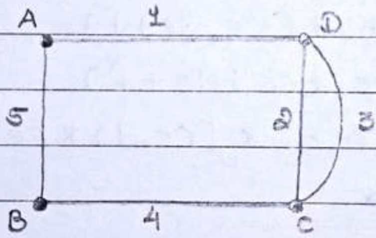
Consider the following graphs.



G_1



G_2



G.

Thus G_1 and G_2 are complements of each other.

Q.4 (a) Show that the algebric structure $(G, *)$ is a group, where $G = \{(a,b) \mid a, b \in R, a \neq 0\}$ and $*$ is a binary operation defined by $(a,b) * (c,d) = (ac, bc+d)$ for all $(a,b), (c,d) \in G$.

Solⁿ :- Let $(a,b), (c,d), (e,f) \in G$; $a \neq 0, c \neq 0, e \neq 0$

1. $(a,b) * (c,d) = (ac, bc+d)$, since $ac \neq 0$, $(ac+bc+d) \in G$.

$\therefore$ G is close under $*$.

2. $[(a,b) * (c,d)] * (e,f) = (ac, bc+d) * (e,f)$

$= (ace, (bc+d)e+f)$

$= (ace, bce+de+f)$

$(a,b) * [(c,d) * (e,f)] = (a,b) * (ce, de+f)$

$= (ace, bce+de+f)$

$\therefore [(a,b) * (c,d)] * (e,f) = (a,b) * [(c,d) * (e,f)]$

$\therefore$ G is associative under $*$.

3. For $(a,b) * (e_1, e_2) = (a,b)$

$\Rightarrow (ae_1 ; be_1 + e_2) = (a,b)$

$\Rightarrow ae_1 = a, \; be_1 + e_2 + b$

$\Rightarrow e_1 = 1, \; e_2 = 0$

$\Rightarrow (e_1, e_2) = (1, 0) \in G$ as $e_1 \neq 0$ is an identity element.

4. For $(a,b) * (c,d) = (e_1, e_2)$

$\Rightarrow (ac, bc+d) = (1, 0)$

$\Rightarrow ac = 1$ and $bc + d = 0$

$\Rightarrow c = 1/a$ and $d = -b/a$

Since $c = 1/a \neq 0$, $(c,d) = \left(\frac{1}{a}, \frac{-b}{a}\right)$ is an inverse of (a,b)

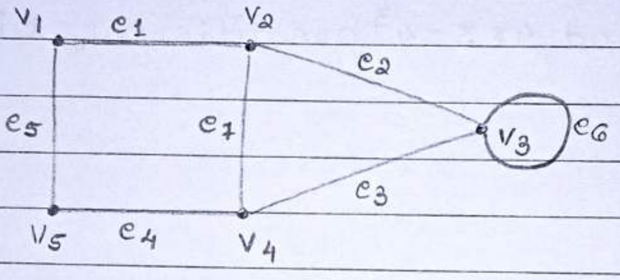
$\therefore (G, *)$ is a group.

(b) Define path and circuit of a graph by drawing the graphs.

Soln:-

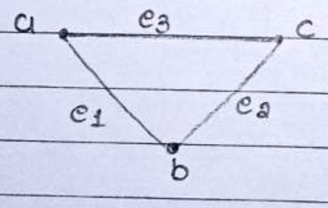
Path: An open walk in which no vertex appears more than once is called a path or an elementary path.

In figure of graph $V_1 e_1 V_2 e_2 V_3 e_3 V_4$ is a path.



Circuit: A closed walk in which no vertex except terminal vertices, is repeated is called a circuit.

In this graph $a e_1 b e_2 c e_3 a$ is a circuit.



(c) (i) Show that the operation $*$ defined by $x * y = x^y$ on the set N of natural numbers is neither commutative nor associative.
(ii) Define ring. Show that the algebraic system $(Z_9, +_9, \cdot_9)$, where $Z_9 = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ under the operations of addition and multiplication of congruence modulo 9, from a ring.

Solⁿ:- (i) Since $3 * 4 = 3^4$ and $4 * 3 = 4^3$ are different, $*$ is not commutative.

Now $(2 * 3) * 4$
$= 2^3 * 4$
$= (2^3)^4$
$= 2^{12}$

and $2 * (3 * 4)$
$= 2 * 3^4$
$= 2^{3^4}$
$= 2^{81}$
$\therefore$ $*$ is not associative.

(ii) Let $a, b, c \in Z_9$
1) obviously $a + b \in Z_9$ as $a+b$ gives the remainder from 0 to 8 on dividing by 9.
2) obviously $a + (b + c) = (a + b) + c$
3) Since $a + 0 = a = 0 + a$, 0 is the identity element.
4) inverse of 0 is 0, 1 is 8, 2 is 7, etc.
5) Since $a + b = b + a$, $+$ is commutative.
6) $a \cdot b$ gives the remainder from 0 to 9 dividing by 9.
$\therefore a \cdot b \in Z_9$

7) obviously $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

8) Again + is distributive over $\cdot$ under congruence modulo m.

$\therefore (Z_9, +_9, \cdot_9)$ is a ring.

Q.5 (a) Define subgraph. Let H be a subgroup of $(Z, +)$, where H is the set of even integers and Z is the set of all integers and + is the operation of addition. Find all right cosets of H in Z.

Solⁿ:- Subgroup: Let $(G, *)$ be a group and S be a non empty subset of G then S is a subgroup of G if itself is a group under the same binary operation $*$.

Since G is a subset of itself, G is itself a subgroup of G. The subset containing only the identity element of a group $(G, *)$ is also a subgroup of $(G, *)$.

We have $H = \{\ldots -4, -2, 0, 2, 4 \ldots\}$

and $Z = \{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}$

For $0 \in Z$, $0 + H = \{\ldots -4, -2, 0, 2, 4 \ldots\}$

For $1 \in Z$, $1 + H = \{\ldots -3, -1, 1, 3, 5 \ldots\}$

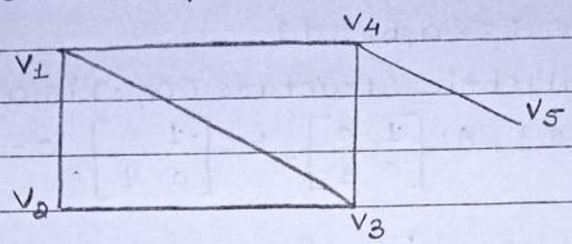
$\therefore$ $0+H$ is same with the left cosets of even integers and $1+H$ is same with the left cosets of odd integers.

$\therefore$ There are only two disinct left cosets of H in Z.

Similarly, $H + 0 = \{\ldots -4, -2, 0, 2, 4 \ldots\}$

and $H + 1 = \{\ldots -3, -1, 1, 3, 5 \ldots\}$ are the distinct right cosets of H in Z.

(b) Define adjacency matrix and find the same for



Soln:- Adjacency matrix :- Let G be a graph with no parallel edges having n vertices and e edges. The adjacency matrix is an $n \times n$ symmetric matrix defined and denoted by

$M(G) = [m_{ij}]$ where

$$m_{ij} = \begin{cases} 1 ; & \text{if } i^{th} \text{ vertex is adjacent to } j^{th} \text{ vertex} \\ 0 ; & \text{otherwise} \end{cases}$$

The adjacency matrix is

$$M = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{c} \begin{array}{ccccc} V_1 & V_2 & V_3 & V_4 & V_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

(c) (i) Draw the composite table for the operation $*$ defined by $x * y = x$, $\forall x, y \in S = \{a, b, c, d\}$

(ii) Show that an algebric structure $(G, \cdot)$ is an abelian group, where $G = \{A, B, C, D\}$, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

and $\cdot$ is the binary operation of matrix multiplication.

Sol^n:- The table is as follows:

$*$	a	b	c	d
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c
d	d	d	d	d

we have $A \cdot B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = B$

similarly, we can find other products as shown in the table

$\cdot$	A	B	C	D
A	A	B	C	D
B	B	A	D	C
C	C	D	A	B
D	D	C	B	A

1. Since all entries in the table are the element of G, G is closed under multiplication.
2. Since associative law holds for matrix multiplication, G is associative under multiplication.
3. From table $A \cdot A = A$, $B \cdot A = B$, $C \cdot A = C$, $D \cdot A = D \Rightarrow A$ is an identity element in G.
4. From table $A \cdot A = A$, $B \cdot B = A$, $C \cdot C = A$, $D \cdot D = A \Rightarrow$ Every matrix is the inverse of itself. Thus inverse exists.
5. From the table we can see that the entries in the rows are same with the entries in corresponding columns.
   Thus G is a commutative under $\cdot$.

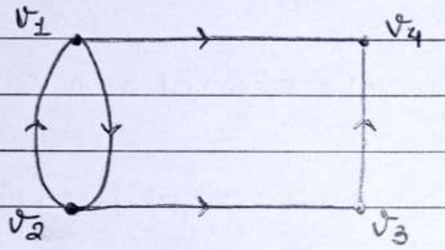
$\therefore (G, \cdot)$ is an abelian group.

Q.6 (a) Define indegree and outdegree of a graph with example.

Sol^n:- In a directed graph G, the indegree of a vertex v is the number of edges ending at v. It is denoted by Indeg (v).

In a directed graph G, the outdegree of a vertex v is the number of edges beginning at v. It is denoted by outdeg (v).

e.g Consider the following directed graph.



Indeg $(v_1) = 1$ outdeg $(v_1) = 2$

Indeg $(v_2) = 1$ outdeg $(v_2) = 2$.

(b) Prove that the inverse of an element is unique in a group $(G, *)$.

Sol^n:- Let G be any group with operation '$*$'.

Let b and c be two ~~identities~~ inverse element of G.

then $\forall a \in G$.

$a * b = e = b * a$ &

$a * c = e = c * a$

~~in particular~~ Now $b * (a * c) = b * c = b$ (e identity)

~~$e * e' = e$~~ ~~(e' is identity)~~ $(b * a) * c = e * c = c$

~~$e * e' = e'$~~ ~~(e is identity)~~ Since G is a group

$b * (a * c) = (b * a) * c$ i.e $b = c$ (associative property)

hence inverse element is unique.

(c) (i) does a 3 regular graph with 5 vertices exist?
(ii) define centre of a graph and radius of a tree?

Solⁿ:- (i) we have $r = 3$, $n = 5$

$\therefore$ Number of edges $= \frac{rn}{2} = \frac{3 \times 5}{2} = \frac{15}{2}$

not an integer number

$\therefore$ Such a graph does not exist.

(ii)
Centre of the graph: A vertex having minimum eccentricity is called the centre of the graph G. That is
$C(G) = \{ v_i \mid v_i$ has minimum eccentricity$\}$

Radius of a tree: The eccentricity of the centre in a tree is called the radius of a tree. That is
$r(G) = \min \{ E(v) \mid$ for all v in $G \}$.

Q.7(a) Check the properties of commutative and associative for the operation $*$ defined by $x * y = x + y - 2$ on the set Z of integers.

Sol^n:- Let $a, b, c \in Z$

i) we have $a * b = a + b - 2$

and $b * a = b + a - 2$

$\therefore a * b = b * a$

$\Rightarrow$ $*$ is commutative

we have $a * (b * c) = a * (b + c - 2)$

$= a + (b + c - 2) - 2$

$= a + b + c - 4$

and $(a * b) * c = (a + b - 2) * c$

$= (a + b - 2) + c - 2$

$= a + b + c - 4$

$\Rightarrow$ $*$ is associative.

(b) define group permutation. Find the inverse of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$.

Sol^n:- i) Let S be a finite set then the mapping $f: S \to S$ is called a permutation of S if f is one-to-one and onto.

if number of elements in S is called the degree of permutation

If S = $\{a_1, a_2 \dots a_n\}$ the the permutation of S is denoted by

$$f = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ f(a_1) & f(a_2) & \dots & f(a_n) \end{pmatrix}$$

ii)

inverse of the permutation is given by

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

(c) (i) Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

(ii) obtain the d.n.f of the form $(p \rightarrow q) \wedge (\sim p \wedge q)$.

Soln:- i)

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

ii) $(p \rightarrow q) \wedge (\sim p \wedge q)$

$\equiv (\sim p \wedge q) \wedge (\sim p \wedge q)$

$\equiv (\sim p \wedge \sim p \wedge q) \vee (q \wedge \sim p \wedge q)$ ($\because$ distributive law)

$\equiv (\sim p \wedge q) \vee (\sim p \wedge \sim q)$ ($\because$ idempotence & commutative law).

Q.8 (a) Find the domain of the function $f(x) = \sqrt{16-x^2}$.

Soln:- (i) For $f(x) = \sqrt{16-x^2}$

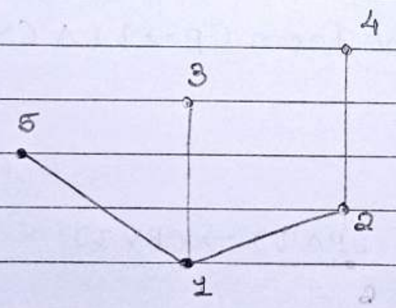
Function is not defined for $16-x^2 \le 0$ $-4 \le x \le 4$

$\therefore$ Function is not defined for $x \in [-4, 4]$

$\therefore D_f = R - [-4, 4]$

(b) Define lattice. Determine whether POSET {{1, 2, 3, 4, 5}} is a lattice.

Soln:- A lattice is a poset $(A, \le)$; if for all $x, y \in A$ then elements lub $\{x, y\}$ and glb $\{x, y\}$ both exists in A.



here there is no upper bounds for 2 and 3

$\therefore$ There is no lub for 2 and 3.

$\therefore$ It is not a lattice.

(c) Show that the propositions $\sim(p \wedge q)$ and $\sim p \vee q$ are logically equivalent.

Soln:- $\sim(p \wedge q)$

p	q	$p \wedge q$	$\sim(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

$\sim p \vee q$

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

* From the truth table value of $\sim(p \wedge q)$ and $\sim p \vee \sim q$ both are equal

i.e $\sim(p \wedge q) \equiv \sim p \vee \sim q$.